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# Laminar convection of a radiating gas in a vertical channel

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An exact solution is obtained for the problem of fully-developed, radiating, laminar convective flow in a vertical heated channel. The effect of radiation is to decrease the temperature difference between the gas and the wall, thereby reducing the influence of natural convection. Thus, the reduction in velocity occurring in a heated upflow is less for a radiating gas. Graphs are presented for the dimensionless velocity and temperature profiles and for the volume and heat fluxes.

## Introduction

Natural convection is the transport of energy resulting from a distributed buoyancy force which, for our study, results from variations in density. The buoyancy force also modifies the flow, thereby causing an interaction between the velocity and temperature fields. This coupling is a fundamental characteristic of natural convection flows and therefore requires the velocity and temperature fields to be solved simultaneously.

Problems of combined natural and forced flows have been studied by Ostrach (1954, 1958). He considered the flow between vertical plates and included the effects of viscous dissipation and heat sources. Morton (1959, 1960) also considered the combined flow problem in vertical as well as in horizontal pipes. In this study we consider the combined natural and forced flow of a radiating gas between vertical plates. The effect of the radiating medium is to further modify the temperature and the velocity fields. We restrict ourselves to the optically thin limit so that the gas emits but only absorbs radiation emitted by the boundaries. This condition is one of physical interest. Furthermore, the study of optically thin flows has, in the past, resulted in increased understanding of the phenomena which could then be utilized in the analysis of more general and more complex problems (cf. Habib & Greif 1970).

## Formulation

We consider the steady fully-developed flow of a fluid between vertical, plane, parallel walls under a constant pressure gradient. It is assumed that the viscosity, thermal conductivity and specific heat are independent of temperature and that

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the essential influence of the variation in density is included in the body force term; that is, in the coefficient of thermal expansion. The temperatures of the walls are the same and are maintained at a constant temperature gradient $\tau/b$ so that the wall temperature, $T_w$, is given by $T_w = T_0 + (\tau/b)x$, where $x$ is the co-ordinate in the vertical direction, $T_0$ is the temperature of the wall at the origin and, following Morton, may be regarded as representative for the region of flow considered. The flow field and the temperature field are symmetrical about the centre-line of the channel, $y = 0$, with the width of the channel equal to $2b$.

Under the conditions specified the equations of continuity, momentum and energy are given by (cf. Morton 1960):

$$\partial u/\partial x = 0, \tag{1}$$

$$-\left(\frac{1}{\rho_0}\frac{\partial p}{\partial x}+g\right)+\nu\frac{\partial^2 u}{\partial y^2}-\beta g(T_w-T) = 0, \tag{2}$$

$$\partial p/\partial y = 0, \tag{3}$$

$$u\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial y^2}-\frac{1}{\rho_0 c_p}\frac{\partial q_R}{\partial y}, \tag{4}$$

where $u$ is the axial velocity, $\alpha$ the thermal diffusivity, $\nu$ the kinematic viscosity, $\beta$ the coefficient of thermal expansion and $q_R$ is the radiative flux. In the optically thin limit, the fluid does not absorb its own emitted radiation; that is, there is no self-absorption, but it does absorb radiation emitted by the boundaries. Under these conditions, we obtain

$$\partial q_R/\partial y = 4K_p\sigma T^4 - 4K_m\sigma T_w^4, \tag{5}$$

where $K_p$ and $K_m$ are the Planck mean and the modified Planck mean absorption coefficients, respectively (Sparrow & Cess 1966). Cogley, Vincenti & Gilles (1968) have shown that in the optically thin limit for a non-grey gas near equilibrium that

$$dq_R/dy = 4(T-T_w)\int_0^\infty k_{\lambda w}(de_{b\lambda}/dT)_w\,d\lambda, \tag{6}$$

which comes directly from (5) where $k_\lambda$ is the absorption coefficient and $e_{b\lambda}$ is the Planck function. Additional simplifications can be made concerning the spectral properties of radiating gases (cf. Tien 1968) but for our purposes these are not necessary. Thus, the energy equation becomes

$$u\tau/\alpha b = d^2\theta/dy^2 + C\theta, \tag{7}$$

where $\theta(y) = T_w - T$ and $C = (4/K_T)\int_0^\infty k_{\lambda 0}(de_{b\lambda}/dT)_0\,d\lambda$, with $K_T$ the thermal conductivity. All quantities have been evaluated at $T_0$ thereby linearizing (7). We therefore limit our study to small variations in the wall temperature. Introducing the transformations $y = bY$, $u = \alpha U/b$ and $\theta = \tau\phi$ reduces (7) to the dimensionless form

$$d^2\phi/dY^2 - F\phi = -U, \tag{8}$$

where $F = b^2C$. The parameter $F$ is a measure of the energy transport due to radiation relative to that due to conduction. In a similar manner, the momentum equation in the $x$ direction, equation (2), becomes

$$d^2U/dY^2 = Ra\phi - \gamma, \tag{9}$$

where $Ra$, the Rayleigh number, equals $g\beta\tau b^3/\nu\alpha$ and $\gamma$ is given by

$$-b^3((1/\rho_0)\,\partial p/\partial x + g)/\nu\alpha$$

(cf. Morton 1960). Note that in the absence of thermal radiation $F = 0$ and in the absence of natural convection $Ra = 0$.

## Solution

Differentiating (8) twice with respect to $Y$ and then eliminating $U$ by using (9) yields

$$\frac{d^4\phi}{dY^4} - F\frac{d^2\phi}{dY^2} + Ra\phi = \gamma. \tag{10}$$

The general solution to (10) for $b_1$ not equal to $b_2$ is of the form†

$$\phi = \gamma/Ra + A\cosh b_1 Y + B\sinh b_1 Y + C\cosh b_2 Y + D\sinh b_2 Y, \tag{11}$$

where

$$b_{1,2} = \tfrac{1}{2}[F \pm (F^2 - 4Ra)^{\frac{1}{2}}] \tag{12}$$

and $A$, $B$, $C$ and $D$ are arbitrary constants. Equation (11) must satisfy the boundary conditions $\phi(1) = 0 = \phi(-1)$ and from (8) we also have that

$$\phi''(1) = 0 = \phi''(-1),$$

where we have made use of the fact that the velocity vanishes at the walls. The solution to the differential equation subject to these boundary conditions is then given by

$$\frac{\phi Ra}{\gamma} = 1 + \frac{b_1^2\cosh b_2 Y}{(b_2^2 - b_1^2)\cosh b_2} - \frac{b_2^2\cosh b_1 Y}{(b_2^2 - b_1^2)\cosh b_1}. \tag{13}$$

For the condition $F^2 - 4Ra < 0$ the solution can be written more conveniently in the following form:

$$\phi Ra/\gamma = 1 - a_1\cosh b_3 Y\cos b_4 Y + a_2\sinh b_3 Y\sin b_4 Y, \tag{14}$$

where

$$b_3 = \tfrac{1}{2}(2Ra^{\frac{1}{2}} + F)^{\frac{1}{2}}, \quad b_4 = \tfrac{1}{2}(2Ra^{\frac{1}{2}} - F)^{\frac{1}{2}} \tag{15}$$

and

$$a_1 = \frac{1}{(4Ra - F^2)^{\frac{1}{2}}}\cdot\frac{F\sinh b_3\sin b_4 + (4Ra - F^2)^{\frac{1}{2}}\cosh b_3\cos b_4}{\cosh^2 b_3\cos^2 b_4 + \sinh^2 b_3\sin^2 b_4}, \tag{16a}$$

$$a_2 = \frac{1}{(4Ra - F^2)^{\frac{1}{2}}}\cdot\frac{F\cosh b_3\cos b_4 - (4Ra - F^2)^{\frac{1}{2}}\sinh b_3\sin b_4}{\cosh^2 b_3\cos^2 b_4 + \sinh^2 b_3\sin^2 b_4}. \tag{16b}$$

For the condition $F^2 = 4Ra$ we obtain the following solution

$$\phi Ra/\gamma = 1 - a_3\cosh EY + a_4 Y\sinh EY, \tag{17}$$

where $E \equiv Ra^{\frac{1}{4}}$ and

$$a_3 2E\cosh^2 E = E^2\sinh E + 2E\cosh E, \tag{18a}$$

$$a_4 2E\cosh^2 E = E^2\cosh E. \tag{18b}$$

† The solution for the case $b_1$ equal to $b_2$ is given in (17).

For the non-radiating problem, $F = 0$, and the temperature distribution becomes

$$\frac{\phi Ra}{\gamma} = 1 - \frac{\cosh[(1+Y)E/2^{\frac{1}{2}}]\cos[(1-Y)E/2^{\frac{1}{2}}] + \cosh[(1-Y)E/2^{\frac{1}{2}}]\cos[(1+Y)E/2^{\frac{1}{2}}]}{\cosh(E2^{\frac{1}{2}}) + \cos(E2^{\frac{1}{2}})}. \quad (19)$$

This relation agrees with the result of Tao (1960) for steady laminar flow in a vertical channel under the influence of combined natural and forced convection. In the absence of both natural convection, $Ra = 0$, and radiation, $F = 0$, the temperature distribution is given by

$$\phi = (\gamma/24)(Y^4 - 6Y^2 + 5). \quad (20)$$

Once we have determined the temperature distribution, the velocity profile can be obtained from (9). Alternatively, the velocity profile could have been determined first thereby permitting the evaluation of the temperature profile. The basic point, of course, is that the presence of natural convection couples the energy and momentum equations.

Substituting the 'general' result for the temperature, that is, (13) into (9) and integrating yields

$$\frac{U(b_2^2 - b_1^2)}{\gamma} = \frac{b_1^2}{b_2^2}\left(\frac{\cosh b_2 Y}{\cosh b_2} - 1\right) - \frac{b_2^2}{b_1^2}\left(\frac{\cosh b_1 Y}{\cosh b_1} - 1\right). \quad (21)$$

From (9) and (14) we obtain for the condition $F^2 - 4Ra < 0$ the result

$$\begin{aligned}\frac{U}{\gamma} = \frac{F}{4Ra} - \frac{2}{4Ra - F^2}\frac{1}{Ra^{\frac{1}{2}}}(\cosh^2 b_3 \cos^2 b_4 + \sinh^2 b_3 \sin^2 b_4)^{-1} \\ \times [(\tfrac{1}{2}F^2 - Ra)(\sinh b_3 \sin b_4 \cosh b_3 Y \cos b_4 Y - \cosh b_3 \cos b_4 \sinh b_3 Y \sin b_4 Y) \\ + \tfrac{1}{2}F(4Ra - F^2)^{\frac{1}{2}}(\sinh b_3 \sin b_4 \sinh b_3 Y \sin b_4 Y \\ + \cosh b_3 \cos b_4 \cosh b_3 Y \cos b_4 Y)]. \quad (22)\end{aligned}$$

From (9) and (17) we obtain for the condition $F^2 = 4Ra$, with $E \equiv Ra^{\frac{1}{4}}$

$$\frac{U}{\gamma} = \frac{2}{E^2}\left[1 - \frac{2\cosh E \cosh EY + \frac{1}{2}E(\sinh E \cosh EY - Y\cosh E \sinh EY)}{2\cosh^2 E}\right]. \quad (23)$$

In the absence of radiation the velocity profile is given by

$$\frac{UE^2}{\gamma} = \frac{\sinh[(1+Y)E/2^{\frac{1}{2}}]\sin[(1-Y)E/2^{\frac{1}{2}}] + \sinh[(1-Y)E/2^{\frac{1}{2}}]\sin[(1+Y)E/2^{\frac{1}{2}}]}{\cosh(E2^{\frac{1}{2}}) + \cos(E2^{\frac{1}{2}})}, \quad (24)$$

which is in agreement with Tao (1960). In the absence of free convection, $Ra = 0$, and we obtain

$$U/\gamma = \tfrac{1}{2}(1 - Y^2). \quad (25)$$

Note that for this case the velocity profile is independent of the temperature profile.

## Results and discussion

The results for the non-radiating problem are presented in figures 1 and 2 and show the effect of the Rayleigh number on the velocity and temperature profiles. In figure 1 the ratio of the dimensionless velocity $U$ to the effective Reynolds number $\gamma$ is plotted against the dimensionless distance from the axis $Y$. The curve for zero Rayleigh number is the parabolic profile for normal Poiseuille

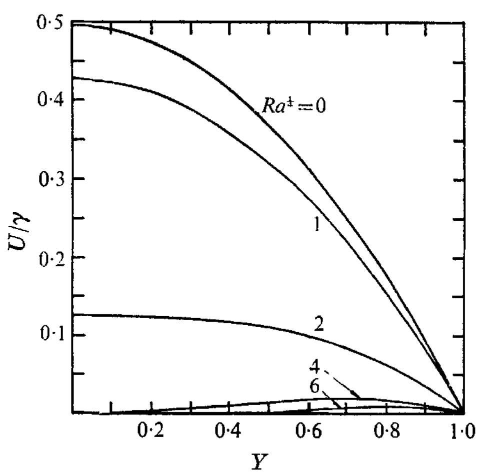


FIGURE 1. Dimensionless velocity profiles for non-radiating problem ($F = 0$).

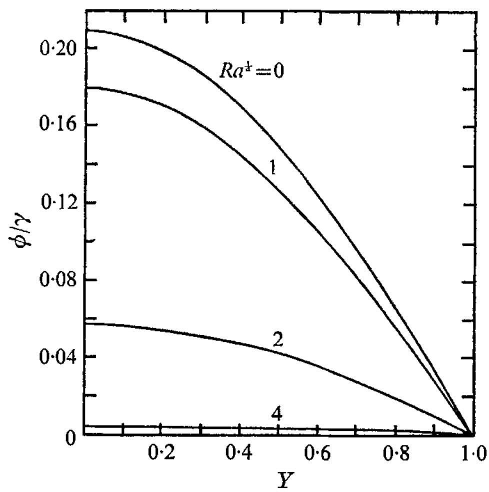


FIGURE 2. Dimensionless temperature profiles for non-radiating problem ($F = 0$).

flow at uniform temperature. For increasing values of the Rayleigh number the velocities decrease over the entire cross-section of the channel with the upflow becoming more concentrated in the region near the wall where the pressure and the buoyancy forces augment one another. The corresponding curves for the dimensionless distributions of the difference between the wall temperature and the temperature of the gas, $\gamma^{-1}\phi$, are shown in figure 2. For increasing values of the Rayleigh number the increased rate of energy transport to the gas is manifested in higher gas stream temperatures or alternatively, in smaller temperature differences. The results are similar to those previously presented by Ostrach and Morton.

The results incorporating the effects of radiation on the velocity and temperature profiles are presented in figures 3 and 4. The effect of radiation is to increase the rate of energy transport to the gas, thereby increasing the temperature of the gas. Thus, for a given value of the Rayleigh number, increasing the radiation parameter $F$ results in the flatter temperature profiles shown in figure 3. The effect of radiation, therefore, is to reduce the influence of natural convection. Recall that the driving force for natural convection is equal to $Ra\phi$. Thus, the velocities for the radiating natural convection condition are greater than those resulting from the corresponding non-radiating natural convection problem (cf. figure 4). Furthermore, for large values of the Rayleigh number the upflow is less concentrated in the region near the walls when radiation is present.

The dimensionless rate of volume flow through the channel per unit of width is given by

$$2\int_0^1 (U/\gamma)\,dY$$

and is plotted against $Ra^{\frac{1}{2}}$ in figure 5. The effect of natural convection is to decrease the volume flux, the influence becoming more pronounced with increasing values of the Rayleigh number. Indeed, there is little effect for small Rayleigh numbers. Thus, for small Rayleigh numbers ($Ra < 1$) the effect of radiation on the flow rate is negligible while for moderate values of the Rayleigh number there is a significant increase in the flow rate resulting from the radiation interaction.

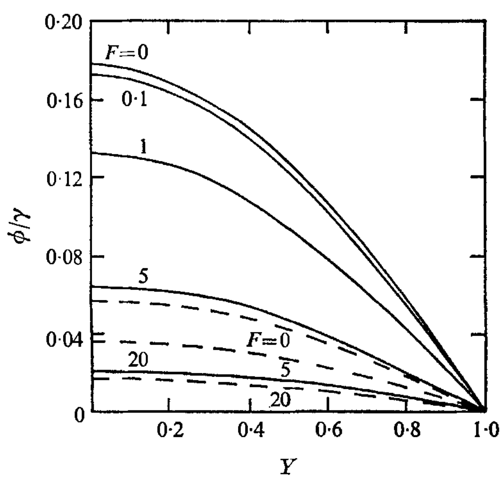


FIGURE 3. Dimensionless temperature profiles including effects of natural convection and radiation. ——, $Ra^{\frac{1}{2}} = 1$; --, $Ra^{\frac{1}{2}} = 2$.

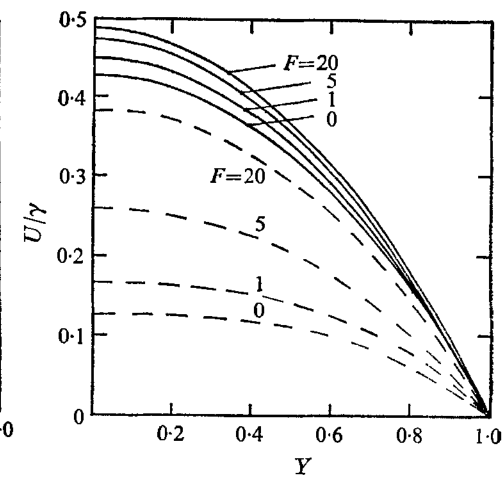


FIGURE 4. Dimensionless velocity profiles including effect of natural convection and radiation. ——, $Ra^{\frac{1}{2}} = 1$; --, $Ra^{\frac{1}{2}} = 2$.

In describing the heat transfer we define the Nusselt number according to $Nu_{\text{tot}} = q_{\text{tot},w} D_h / K_T \theta_m$, where $q_{\text{tot},w}$ is the total heat flux at either wall, $D_h$ is the hydraulic diameter which equals twice the channel width ($4b$) and $K_T$ is the thermal conductivity. The difference between the wall temperature and the mean temperature across the channel, $\theta_m = T_w - T_m$, is given by $\int_0^1 \theta\, dY$. The results for the Nusselt number are presented in figure 6. The effect of natural convection is to increase the heat transfer rate although only slightly for Rayleigh numbers less than 40. Now, however, the effect of radiation is of importance for all Rayleigh numbers and causes a dramatic increase in the heat transfer rate.

The Nusselt number may be written as follows:

$$Nu_{\text{tot}} = Nu_{\text{rad}} + (Nu_{\text{cond}})_{\text{with rad present}}, \tag{26}$$

where the first term in (26) is the contribution by radiation and the second term is the contribution by conduction (and natural convection) with radiation present. The radiative flux at the wall may be obtained by integrating (6) yielding the result

$$q_{\text{rad, wall}} = bCK_T(T_w - T_m), \tag{27}$$

so that $Nu_{\text{rad}}$ equals $4F$. The fact that the optically thin radiative flux is proportional to $T_w - T_m$ and therefore yields a value for $Nu_{\text{rad}}$ that is invariant with respect to axial distance was previously noted by Greif & McEligot (1970)

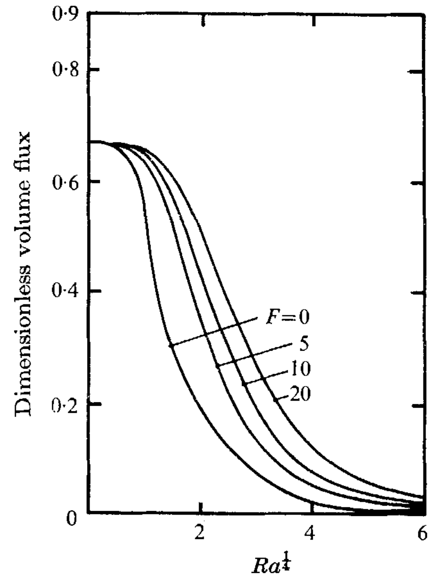


FIGURE 5. Dimensionless volume flux including effects of natural convection and radiation.

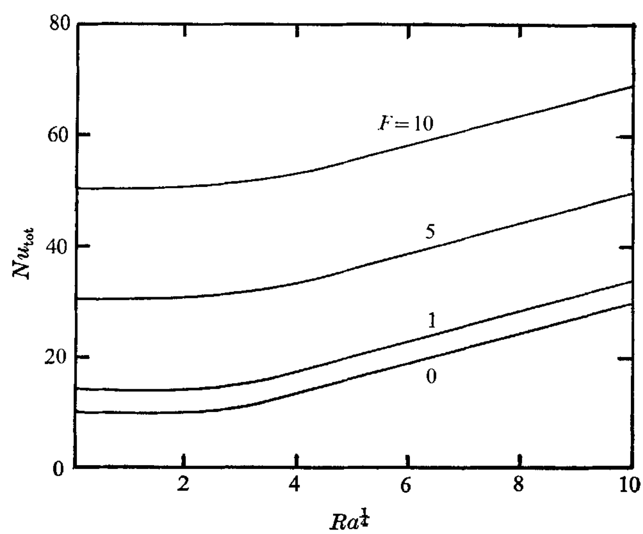


FIGURE 6. Dimensionless heat flux including effects of natural convection and radiation.

for a constant wall temperature problem. Recall that the present problem is restricted to small variations in the wall temperature. The contribution to the Nusselt number by conduction and natural convection may be directly obtained from the relation:

$$(Nu_{\text{cond}})_{\text{with rad present}} = \left(\frac{\partial \phi}{\partial Y}\right)_w \frac{4}{\phi_m}, \qquad (28)$$

where the temperature profiles, $\phi(Y)$, are given in (13), (14) and (19). From the results shown in figure 6 it is clear that (28) is only slightly affected by radiation. Thus, a very good approximation for the contribution by conduction and natural convection to the Nusselt number with radiation present is to simply use the result obtained for the non-radiating problem ($F = 0$) so that

$$(Nu_{\text{cond}})_{\text{with rad present}} \cong (Nu_{\text{cond}})_{\text{non-rad}}.$$

Thus, the total Nusselt number (cf. (26)) is given to a very good approximation by

$$Nu_{\text{tot}} \cong 4F + \frac{2G(\sinh G - \sin G)}{\cosh G + \cos G - (\sinh G + \sin G)/2G}, \tag{29}$$

where $G = 2^{\frac{1}{2}} Ra^{\frac{1}{4}}$.

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